

Arbitrary large number of non trivial rescaling limits

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Abstract

We construct a family of rational map sequences providing an arbitrary large number of independent rescaling limits of non monomial type. From this, we deduce the existence of a family of rational maps providing a non trivial dynamics on the Berkovich projective line over the field of formal Puiseux series.

1 Introduction.

Let us denote by $\mathbb{S} := P^1(\mathbb{C})$ the Riemann sphere. In this paper we are interested in the behavior of the elements of Rat_d the set of rational maps $f : \mathbb{S} \rightarrow \mathbb{S}$ of exact degree d under the iteration by composition. More precisely, we are interested in the phenomena of existence of rescaling limits detected in [Sti] but defined for the first time in [Kiw15] as follows.

Definition. *For a sequence of rational maps $(f_n)_n$ in Rat_d , a rescaling is a sequence of Moebius transformations $(M_n)_n$ such that there exist $k \in \mathbb{N}$ and a rational map g of degree ≥ 2 such that*

$$M_n \circ f_n^k \circ M_n^{-1} \rightarrow g$$

uniformly on compact subsets of \mathbb{S} with finitely many points removed.

If this k is minimal then it is called the rescaling period for $(f_n)_n$ at $(M_n)_n$ and g a rescaling limit for $(f_n)_n$.

In that paper Jan Kiwi introduced the notions of independence of rescalings and of dynamical dependence of rescalings below (these notions are not used in this paper).

Definition (Independence of rescalings). *We say that two rescalings $(M_n)_n$ and $(N_n)_n$ of a sequence of rational maps $(f_n)_n$ are independent and write $N_n \sim M_n$ if*

$$N_n \circ M_n^{-1} \rightarrow \infty$$

in Rat_1 . That is, for every compact set K in Rat_1 , the sequence $N_n \circ M_n^{-1} \notin K$ for n big enough. The rescalings are said to be equivalent if $N_n \circ M_n^{-1} \rightarrow M$ in Rat_1 .

Definition (Dynamical dependence of rescalings). *Given a sequence $(f_n)_n \in \text{Rat}_d$ and given $(M_n)_n$ and $(N_n)_n$ of period dividing q . We say that $(M_n)_n$ and $(N_n)_n$ are dynamically dependent if, for some subsequences $(M_{n_k})_{n_k}$ and $(N_{n_k})_{n_k}$, there exist $1 \leq m \leq q$, finite subsets S_1, S_2 of \mathbb{S} and non constant rational maps g_1, g_2 such that*

$$L_{n_k}^{-1} \circ f_{n_k}^m \circ M_{n_k} \rightarrow g_1$$

uniformly on compact subsets of $\mathbb{S} \setminus S_1$ and

$$M_{n_k}^{-1} \circ f_{n_k}^{q-m} \circ L_{n_k} \rightarrow g_2$$

uniformly on compact subsets of $\mathbb{S} \setminus S_2$.

Jan Kiwi proved the two following results.

Theorem A. *[Kiw15] For every sequence in Rat_d for $d \geq 2$, there are at most $2d - 2$ classes of dynamically independent rescalings with a non postcritically finite rescaling limit.*

Theorem B. *[Kiw15] For every sequence in Rat_2 , there are at most 2 classes of dynamically independent rescalings.*

Following these result, J.Kiwi wrote in [Kiw15] the following natural question:

Question 1.1. *Is there a bound on the number of classes of dynamically independent and non-monomial rescalings that a sequence in Rat_d can have that would depend only on the degree d ?*

In this paper we prove that the answer is no. More precisely we prove the following theorem:

Theorem 1. *For every $n \in \mathbb{N}^*$ and $d \geq 3$ there exists a sequence of rational maps $(f_k)_k$ in Rat_d with n dynamically independent, post-critically finite and non-monomial rescaling limits.*

Theorem A and Theorem B have been proven using non-archimedean dynamics tools and later re-proven in [Arfa] and [Arfb] respectively, using a different approach based on the Deligne-Mumford compactification of the moduli space of stable punctured spheres. This compactification has been restated with the language of trees of spheres in [Arfc]. We suppose that the reader has a good knowledge of the definition provided in [Arfa]. With this vocabulary, Theorem 1 can be written the following way:

Theorem 2. *For every $n \in \mathbb{N}^*$ and $d \geq 3$ there exists a dynamical system between trees of spheres of degree d limit of dynamically marked rational maps, with n cycles of spheres whose associated cover is post-critically finite and non-monomial.*

In fact these two approaches are related and their relation has been sketched in [Arfd]. The results of [Arfa] and [Arfb] are a translation of results in [Kiw15]. The authors use dynamical systems between trees of spheres in one case or dynamics on Berkovich spaces in the other, in order to deduce results in holomorphic dynamics. The approach here is the exact reverse.

We are inspired by the holomorphic dynamical notion of “Shishikura trees”. Those have been introduced by M. Shishikura in [Shi89] and [Shi02] for a special case and developed as a general idea during his talks. Following these ideas, [CP] define another notion of Shishikura trees that we will use for our purpose. Our result will be based on the constructions of self-graphing given in [CP].

Denote by \mathbb{D}^* the punctured unit disc of \mathbb{C} . In the analytic context, rescalings, rescaling limits, independence of rescalings, dynamical dependance of rescaling limits are also defined by replacing $n \in \mathbb{N}$ tending to infinity by $t \in \mathbb{D}^*$, $n \rightarrow \infty$ by $t \rightarrow 0$ in the previous definitions, and requiring holomorphic dependence on t (see [Arfd]). Jan Kiwi proved the following.

Proposition 1.2 ([Kiw15]Proposition 6.1). *Consider a sequence of degree d rational maps f_n . Let $N \in \mathbb{N}$ and assume that for all $j = 1, \dots, N$ the sequence $(M_{j,n})_n$ is a rescaling of period q_j for $(f_n)_n$ with rescaling limit g_j . Then there exists a degree d holomorphic family $(f_t)_t$ and, for each $j = 1, \dots, N$, a holomorphic family of Moebius transformations $(M_{j,t})_t$ such that $(M_{j,t})_t$ is a rescaling for $(f_t)_t$ of period q_j and limit g_j .*

If $(M_{j,t})_t$ and $(M_{k,t})_t$ are dynamically dependent for $(f_t)_t$, then $(M_{j,n})_n$ and $(M_{k,n})_n$ are dynamically dependent for $(f_n)_n$.

Hence Theorem 1 has the following consequence.

Corollary 1. *For every $n \in \mathbb{N}^*$ and $d \geq 3$ there exists a holomorphic family $f_t \in \text{Rat}_d$ for $t \in \mathbb{D}^*$ with n dynamically independent, post-critically finite and non-monomial rescaling limits.*

Using the bridge between non-archimedean dynamics and the dynamics on trees of spheres made explicit in [Arfd], we deduce Corollary 2 below.

Denote by \mathbb{L} the completion of the field of formal Puiseux series over \mathbb{C} equipped with its usual non-archimedean norm. Denote by $\mathbb{P}_{\text{Berk}}^1$ the Berkovich projective line over \mathbb{L} and recall the type II points of $\mathbb{P}_{\text{Berk}}^1$ are the points separating at least three different branches.

Corollary 2. *For every $n \in \mathbb{N}^*$ and $d \geq 3$ there exists $f \in \mathbb{L}(Z)$ of degree d with n periodic type II points in disjoint cycles for the induced dynamics in $\mathbb{P}_{\text{Berk}}^1$ whose reduction is post-critically finite and non-monomial.*

We will not recall more details about the non-archimedean dynamics that the interested reader can find in [Kiw15] (or [Arfd] for a first reading).

Outline.

In Section 2, we recall the notion of Shishikura trees from [CP] with some small adaptations. Section 3 deals with the self-grafting construction. In Section 4, we explicit the relation between Shishikura trees and dynamical systems between trees of spheres. In Section 5, we prove Theorem 1 and Theorem 1. The article ends on a little appendix with some technical results on stable trees.

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2 Construction of a Shishikura tree

In this section we recall the constructions and results from [CP] with a few changes on some definitions. We point out some properties needed in the next sections.

2.1 Canonical multicurve

Here f denote a hyperbolic rational map. We denote by \mathcal{P}_f its post-critical set and \mathcal{P}'_f the set of accumulation points of \mathcal{P}_f .

Suppose that $E \subset \mathbb{S}$ is a connected set which is neither open or closed. We say that E is **disc-type** if E is contained in a disk D with $\text{card } D \cap \mathcal{P}_f \leq 1$. We say that E **annular-type** if E is not disk-type and is contained in an annulus A with $\text{card } A \cap \mathcal{P}_f = 0$. If E is neither disk-type nor annular-type, then we say that E is **complex-type**.

Recall that given a set $X \subset \mathbb{S}$, a **multicurve** on $\mathbb{S} \setminus X$ is a collection of Jordan curves γ in $\mathbb{S} \setminus X$ pairwise disjoint and non isotopic and non-peripheral (i.e. every connected component of $\mathbb{S} \setminus \gamma$ contains at least two points of \mathcal{P}_f). A multicurve Γ_f is **totally stable** (by f) if each non-peripheral curve of $f^{-1}(\gamma)$ for $\gamma \in \Gamma_f$ is isotopic rel \mathcal{P}_f to a curve in Γ_f and if each curve $\gamma \in \Gamma_f$ is isotopic rel \mathcal{P}_f to a curve in $f^{-1}(\gamma')$ for some curve $\gamma' \in \Gamma_f$.

Theorem 2.1. *There exist a totally stable multicurve Γ_f in $\mathbb{S} \setminus \mathcal{P}_f$ such that $\forall D \in \mathbb{S} \setminus \Gamma_f$, D contains a unique complex type Julia or is contained in a complex-type Fatou domain, and conversely.*

These properties are stable under isotopy rel \mathcal{P}_f and the isotopy class of such a curve is called the **canonical multicurve** of f . As a direct consequence of these definitions we have the following property:

Lemma 2.2. *Let Γ_f be a representative of the canonical multicurve and U be a connected component of $\mathbb{S} \setminus \Gamma_f$. Denote by $cc\partial U$ the collection of connected components of $\mathbb{S} \setminus \partial U$. Then we have:*

- $\forall V \in cc\partial U, \text{card } V \cap \mathcal{P}_f \geq 1$ and
- $\text{card}(cc\partial U) + \text{card}(U \cap \mathcal{P}_f) \geq 3$.

Given a rational map f , Jordan curve is called a KB curve if it is a connected component of some Koenig or Böttcher coordinates level curve (cf [Mil06] for the definitions of the Koenigs and Böttcher coordinates). Similarly, a multicurve is KB if it consists of KB curves. By construction in [CP], we have:

Proposition 2.3. *Every canonical multicurve has a representative consisting of a KB multicurve.*

Such a representative is called a **KB canonical multicurve** of f .

2.2 Shishikura trees

In this section, Γ_f denotes a KB canonical multicurve of f (cf Proposition 2.3).

Given a collection Γ of disjoint Jordan curves on \mathbb{S} , we define the dual tree \mathcal{T}_Γ of Γ to be the tree whose vertices consist of the connected components of $\mathbb{S} \setminus \Gamma$ and whose edges are the elements of Γ and join two vertices if and only if it lies in both of their boundary.

In [CP], the **Shishikura tree** \mathcal{T}_f is defined to be the dual tree of Γ_f . Let $\Gamma_f^{-1} := f^{-1}(\Gamma_f)$ and \mathcal{T}_f^{-1} be its associated dual tree. Define $\mathcal{P}_f^{-1} := f^{-1}(\mathcal{P}_f)$. The map f induces a tree map $\tau_f : \mathcal{T}_f^{-1} \rightarrow \mathcal{T}_f$ (indeed, it maps adjacent vertices to adjacent vertices and edges to edges, cf Figure 1).

Remark 2.4. *This map is slightly different than the one in [CP] where the authors consider Γ_1 the subset of Γ_f^{-1} consisting of its non peripheral elements in $\mathbb{S} \setminus \mathcal{P}_f$ to define a tree \mathcal{T}_1 and then a map $\tau_1 : \mathcal{T}_1 \rightarrow \mathcal{T}_f$. There is a natural map $\pi_1 : \mathcal{T}_f^{-1} \rightarrow \mathcal{T}_1$ consisting in the inclusion. The relation between τ_f and τ_1 is provided by the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{T}_f^{-1} & \xrightarrow{\pi_1} & \mathcal{T}_1 \\ & \searrow \tau_f \quad \swarrow \tau_1 & \\ & \mathcal{T}_f & \end{array}$$

and this change of definition, necessary here, does not affect the results cited from [CP].

With our definition, the map f induces also a degree function on the set of edges and vertices that we will denote by \deg_f . From the total stability of Γ_f we deduce a natural identification of \mathcal{T}_f in \mathcal{T}_f^{-1} and the map τ_f induces dynamics on \mathcal{T}_f .

3 Self-grafting

In [CP] the authors constructed a sequence of rational maps $(f_n)_n$ using **self-grafting**. We recall here what this procedure allowed to construct.

In [God], Sébastien Godillon considered the following family of rational maps

$$f_0(z) := \frac{(1 - \lambda)[(1 - 4\lambda + 6\lambda^2 - \lambda^3)z - 2\lambda^3]}{(z - 1)^2[(1 - \lambda - \lambda^2)z - 2\lambda^2(1 - \lambda)]}$$

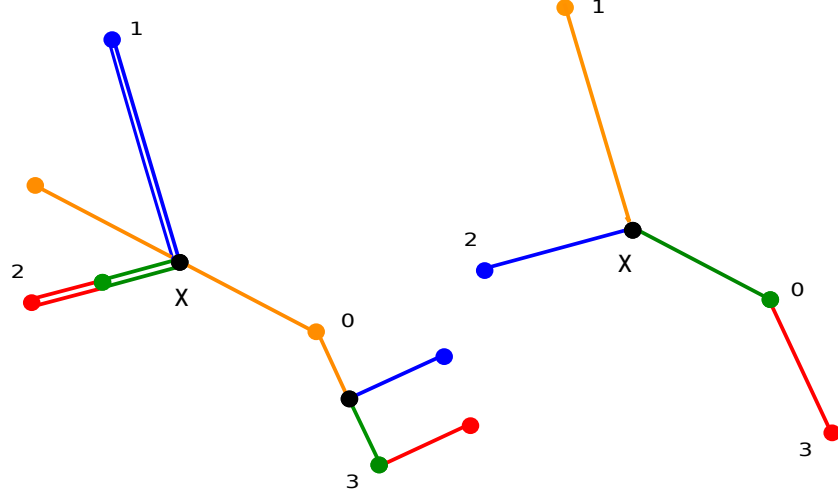


Figure 1: The Shishikura trees $\mathcal{T}_{f_0}^{-1}$ (left) and \mathcal{T}_{f_0} (right) for the function f_0 introduced in Section 3. The map τ_f maps the vertices and edges to the ones of identical colors. The vertex x is fixed and the numbered vertices form a cycle of period 4.

for $\lambda \neq 0, 1$. We fix some parameter $\lambda_0 \neq 0, 1$ and denote by f_0 the corresponding element in this family. Figure 1 shows the Shishikura trees $\mathcal{T}_{f_0}^{-1}$ and \mathcal{T}_{f_0} . In order to simplify the notations let us respectively denote \mathcal{T}_{f_0} and τ_{f_0} by \mathcal{T}_0 and τ_0 .

The self-grafting is an induction procedure that from some Shishikura tree \mathcal{T}_n uses Thurston's realization theorem to construct

- a new tree \mathcal{T}_{n+1} ,
- a map τ_{n+1} acting on it, and
- a hyperbolic rational map f_{n+1} such that $\mathcal{T}_{n+1} = \mathcal{T}_{f_{n+1}}$ and $\tau_{n+1} = \tau_{f_{n+1}}$.

In addition we have $\mathcal{T}_n \triangleleft \mathcal{T}_{n+1}$ in the sense of the following definition.

Definition 3.1. *We say that a tree T^X is compatible with a tree T^Y and write $T^X \triangleleft T^Y$ if*

- *the vertices of T^X are vertices of T^Y and*
- *for all vertices v, v_1, v_2 and v_3 of T^X , the vertex v separates v_1, v_2 and v_3 in T^X if and only if it does the same in T^Y .*

(This definition can be seen as the combinatorial version of the notion of “subtree” for the embedded real trees.)

Hence edges of \mathcal{T}_n between two vertices can be identified as the arc of vertices and edges separating the two corresponding vertices in \mathcal{T}_{n+1} (cf Figure 2). As a consequence $\mathcal{T}_0 \triangleleft \mathcal{T}_n$.

Let us recall here some relations between the tree \mathcal{T}_{n+1} and \mathcal{T}_n . We denote by x_0 and x_1 the two vertices of the \mathcal{T}_0 represented on Figure 2. As $\mathcal{T}_0 \triangleleft \mathcal{T}_n$, the vertices x_0 and x_1 are also vertices of \mathcal{T}_n . We will denote by $[x_0, x_1]_n$ the arc in \mathcal{T}_n joining those. The vertices of \mathcal{T}_{n+1} lying in the edges of \mathcal{T}_n form a cycle of period $k > 2$. Let $v_0 \in \mathcal{T}_{n+1}$ be the vertex on $[x_0, x_1]_{n+1} \setminus \{x_0\}$ which is the closest to x_0 . Denote by B_0 the branch of \mathcal{T}_{n+1} at v_0 that contains the vertex x_0 (by the choice of v_0 , this branch contains only vertices of \mathcal{T}_n). The tree \mathcal{T}_{n+1} is obtained from its elements identified with \mathcal{T}_n by attaching at each point $v_{i+1} := \tau_{n+1}^i(v_i)$ for $i = 0..k-1$ a copy of B_0 that we denote by B_{i+1} .

Denote by $\iota_{n+1} : \mathcal{T}_{n+1} \rightarrow \mathcal{T}_{n+1}$ the involution that exchanges the two branches B_0 and B_k via their natural identification. We extend the map τ_n on \mathcal{T}_{n+1} into a map f_{n+1} that maps B_k to B_1 and B_i to B_{i+1} for $i = 1..k-1$ via the natural identification. The map τ_{n+1} is defined to be $\iota_{n+1} \circ f_{n+1}$. The degree of τ_{n+1} is the one of τ_0 on its elements identified with \mathcal{T}_0 and it is 1 elsewhere.

Remark 3.2. *With these properties we can point out in particular that*

- $v_0 \in [x_0, x_1]_{n+1}$ is k periodic for the map τ_{n+1} with some $k > 1$,
- τ_{n+1}^k maps B_k with degree 1 to B_0 , and
- τ_{n+1} maps B_0 with degree 2 to its image.

4 Jumping between trees

In this section f denotes a hyperbolic rational map. We consider a Shishikura tree \mathcal{T}_f and denote by $\tau_f : \mathcal{T}_f^{-1} \rightarrow \mathcal{T}_f$ the associated Shishikura tree map. We explain how we associate to τ_f a dynamical system between trees of spheres which is a limit of dynamically marked rational maps in the sense of [Arfa].

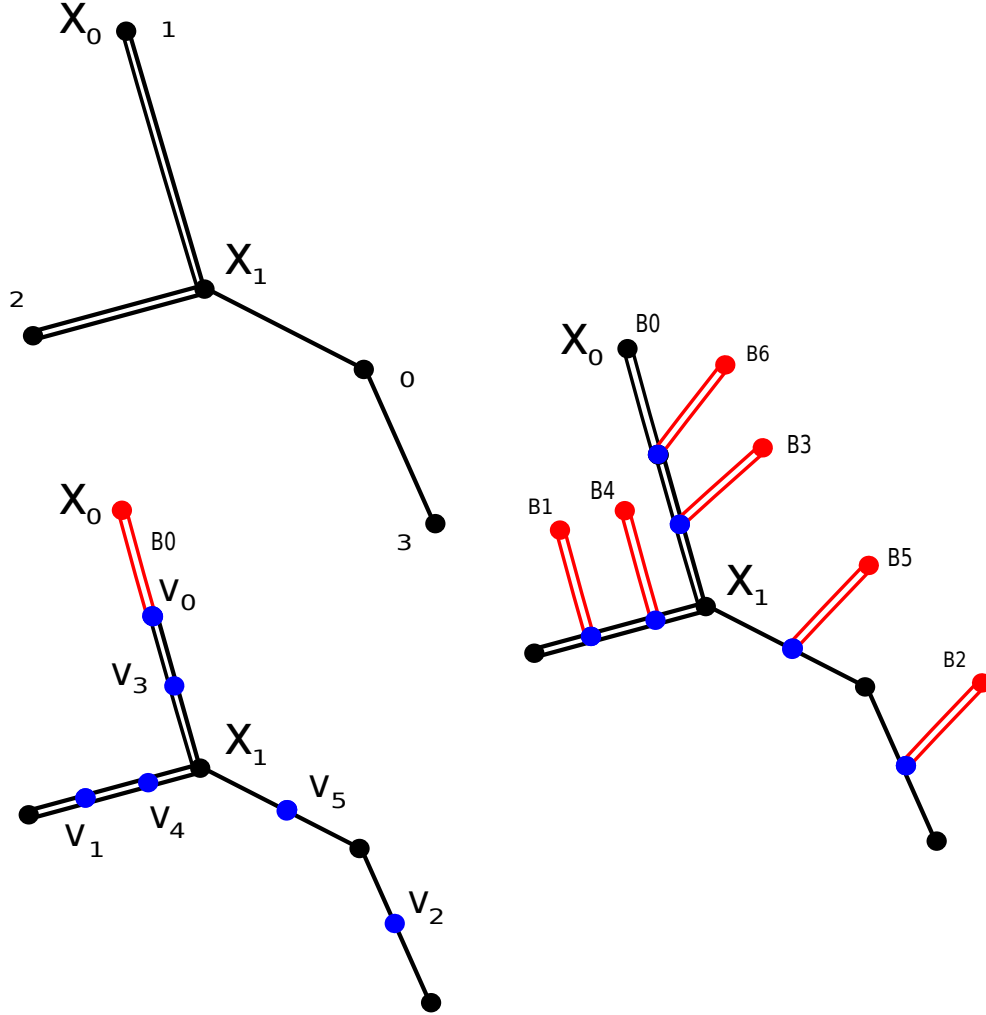


Figure 2: On the top left is the tree \mathcal{T}_{f_0} with its cycle as on Figure 1. The tree lower left is the tree \mathcal{T}_{f_1} restricted to its elements identified with \mathcal{T}_{f_0} and in red is represented the branch B_0 as labeled in Section 3. The v_i form a cycle. On the right is the tree \mathcal{T}_{f_1} and in red are the copies of the branch B_0 from the previous tree.

4.1 Notation for trees of spheres

A tree of spheres marked by a finite set X is usually denoted by \mathcal{T}^X . It is a combinatorial tree that we denote by T^X whose set of leaves is X together with the data for each internal vertex $v \in T^X$ of a conformal sphere denoted by \mathbb{S}_v and a different attaching point $i_v(e) \in \mathbb{S}_v$ for every edge e adjacent to v . We denote by X_v the set of attaching points of edges on \mathbb{S}_v .

If \mathcal{T}^Y is another tree of spheres, we write $\mathcal{T}^X \triangleleft \mathcal{T}^Y$ if $T^X \triangleleft T^Y$ and if the identified edges are attached at the same place.

A cover $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ between two trees of spheres marked by finite sets Y and Z is the following data

- a map $F : T^Y \rightarrow T^Z$ mapping leaves to leaves, internal vertices to internal vertices, and edges to edges,
- for each internal vertex v of T^Y , an holomorphic ramified cover $f_v : \mathbb{S}_v \rightarrow \mathbb{S}_{F(v)}$ that satisfies the following properties:
 - the restriction $f_v : \mathbb{S}_v - Y_v \rightarrow \mathbb{S}_{F(v)} - Z_{F(v)}$ is a cover,
 - $f_v \circ i_v = i_{F(v)} \circ F$,
 - if e is an edge between v and v' , then the local degree of f_v at $i_v(e)$ is the same as the local degree of $f_{v'}$ at $i_{v'}(e)$.

For such a \mathcal{F} , we say that $(\mathcal{F}, \mathcal{T}^X)$ is dynamical system between trees of spheres if $\mathcal{T}^X \triangleleft \mathcal{T}^Y$ and $\mathcal{T}^X \triangleleft \mathcal{T}^Z$.

4.2 Construction of a tree of spheres

4.2.1 Marking sets

It follows from Lemma 2.2 that one can associate to every vertex v of \mathcal{T}_f a set $X_v \subset \mathcal{P}_f$ of cardinal 3 such that two points of X_v lie in the same connected component U of $\mathbb{S} \setminus \Gamma_f$ if and only if $U = v$. Define X to be the union (maybe not disjoint) of the X_v and the set of critical values of f . Let $Z := X \cup f(X)$ and let $Y := f^{-1}(Z)$. Note that by choice we have $X \subset Y \cap Z$.

We define \mathbf{F} to be the pair consisting of $F := (f|_Y : Y \rightarrow Z)$ and the degree function $\deg_F : Y \rightarrow \mathbb{N}$ corresponding to the local degree of $f|_Y$. This \mathcal{F} is a portrait in the sense of [Arfa].

Denote by i and j the respective inclusions of Y and Z into \mathbb{S} . The map (f, i, j) is a rational map dynamically marked by (\mathbf{F}, X) in the sense of [Arfa], i.e. y and z are injective and we have the following commutative diagram :

$$\begin{array}{ccccc} X & \longrightarrow & Y & \xrightarrow{i} & \mathbb{S} \\ & \searrow & \downarrow F & & \downarrow f \\ & & Z & \xrightarrow{j} & \mathbb{S} \end{array}$$

with $\deg_f(i(y)) = \deg_F(y)$ for all $y \in Y$ and $i|_X = j|_X$.

4.2.2 Stretching

Recall that given a set $X' \subset \mathbb{S}$, any disjoint union of curves $\Gamma' \subset \mathbb{S} \setminus X'$ gives a natural partition of X' that corresponds to the set of non empty intersections of X' with the different connected components of $\mathbb{S} \setminus \Gamma'$, we denote it by $\mathcal{P}_{\Gamma', X'}$.

For every $\gamma \in \Gamma$, let us fix an open annulus A^γ in $\mathbb{S} \setminus X$ that retracts to γ and made of KB level curves. Let

$$\mathbb{C}_r := \{c \in \mathbb{C} \mid \Re(c) > 0\}.$$

Proposition 4.1. *There exists a holomorphic family of rational maps f_t for $t \in \mathbb{C}_r$ and a holomorphic motion*

$$\Phi = \left\{ \begin{array}{ccc} \mathbb{C}_r \times \mathbb{S} & \rightarrow & \mathbb{S} \\ (t, z) & \mapsto & \Phi_t(z) \end{array} \right. \quad \text{such that} \quad \begin{array}{ccc} \mathbb{S} & \xrightarrow{f} & \mathbb{S} \\ \Phi_t \downarrow & & \downarrow \Phi_t \\ \mathbb{S} & \xrightarrow{f_t} & \mathbb{S}, \end{array}$$

- $\mathcal{P}_{\Phi_t(\Gamma), \Phi_t(X)} = \mathcal{P}_{\Gamma, X}$, and
- for every $\gamma \in \Gamma_f$, $\text{Modulus}(\Phi_t(A^\gamma)) \rightarrow \infty$ when $t \rightarrow 0$.

Proof. This construction uses the standard "stretching" deformation described for example in [BF13]. For our purpose, we choose a fundamental annulus in each of the grand orbit of the Fatou components that contain one of the KB curve. By stretching the complex structure on these annuli by

making them going to infinity in modulus when t approaches 0, we get such a family of rational maps f_t and a holomorphic motion:

$$\Phi = \left| \begin{array}{l} \mathbb{C}_r \times \mathbb{S} \rightarrow \mathbb{S} \\ (t, z) \mapsto \Phi_t(z) \end{array} \right.$$

such that

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{f} & \mathbb{S} \\ \Phi_t \downarrow & & \downarrow \Phi_t \\ \mathbb{S} & \xrightarrow{f_t} & \mathbb{S}. \end{array}$$

We choose for every $\gamma \in \Gamma_f$, an open annulus A^γ included in the Fatou set of f and that contain γ . By the stretching deformation gives that $\Phi_t(A^\gamma)$ is still an annulus whose modulus tends to infinity as $t \rightarrow 0$. Moreover, the f_t are holomorphic so the postcritical set moves holomorphically with the parameter t so the sumption about the partitions follows. \square

We define such a Φ and take a sequence $t_n \in \mathbb{C}_r$ converging to 0. Then according to [Arfc](Theorem 3), after passing to a subsequence, we have the following:

Proposition 4.2. *The sequence $(f_{t_n}, \Phi_{t_n} \circ i, \Phi_{t_n} \circ j)$ converges dynamically to a dynamical system between trees of spheres $(\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z, \mathcal{T}^X)$ marked by \mathbf{F} .*

4.3 From Shishikura trees to trees of spheres

Define T_f to be the combinatorial tree of \mathcal{T}_f with the additional vertices X and edges between any element $x \in X$ and the vertex corresponding to the connected component of $\mathbb{S} \setminus \Gamma_f$ containing x . Similarly, from \mathcal{T}_f^{-1} we construct T_f^{-1} whose set of leaves is $f^{-1}(X)$. It is clear from the definitions that we have $\mathcal{T}_f \triangleleft T_f$ and $\mathcal{T}_f^{-1} \triangleleft T_f^{-1}$.

In this section we prove the following statement:

Theorem 4.3. *After changing the labeling of the internal vertices of T_f we have $T_f = T^X$ and $T_f^{-1} \triangleleft T^Y$. The map $F : T^Y \rightarrow T^Z$ restricted to T_f is the map τ_f and for every attaching point z of an edge $e \in T^X$ on a sphere associated to a vertex $v \in T^X$, the local degree of $f|_v$ at z is the degree of τ_f at e .*

4.3.1 Stable trees and compatibility

A tree is **stable** if any internal vertex is adjacent to at least three edges.

Remark 4.4. *As a direct consequence of Lemma 2.2, the trees T_f and T_f^{-1} are stable. Also according to [Arfc], the combinatorial trees T^X, T^Y and T^Z associated respectively to $\mathcal{T}^X, \mathcal{T}^Y$ and \mathcal{T}^Z of Proposition 4.2 are stable.*

In this subsection we prove the following lemma.

Lemma 4.5. *If T_1 and T_2 are two stable trees sharing the same set of leaves and such that $T_1 \triangleleft T_2$, then $T_1 = T_2$.*

For any choice of three different leaves in a tree, there exists a unique vertex of this tree separation them. Given an internal vertex v of a combinatorial tree T and a subset X_0 of leaves, there is a natural partition of X_0 associated to v consisting of the collections of non empty intersections of X_0 with the different connected components of $T \setminus \{v\}$.

Lemma 4.6. *Let T_1 and T_2 be combinatorial stable trees. Denote by X_1 the set of leaves of T_1 . Suppose that X_1 is included in the set of leaves of T_2 and that for every $x_1, x_2, x_3 \in X_1$ distinct, the partitions of X_1 associated to the vertices separating x_1, x_2, x_3 are the same. Then, after relabeling the internal vertices of T_1 or T_2 , we have $T_1 \triangleleft T_2$.*

Proof. Take an internal vertex u of T_1 . Then, by stability, this vertex separate three distinct elements x_1, x_2, x_3 of X_1 . Consider the vertex v of T^2 separating the same elements. Then, the vertices are considered modulo relabeling, we can suppose that $u = v$. According to these hypothesis, this relabeling can be made consistently on for all the triples of X_1 , so for all the vertices of T_1 . It follows that we can consider that the vertices of T_1 are vertices of T_2 .

Consider four vertices u, u_1, u_2, u_3 of T_1 . Suppose that u separates u_1, u_2 and u_3 in T_1 . Chose an element x_1 (resp. x_2, x_3) in the branch on v containing v_1 (resp. v_2, v_3). Then, in the tree T_2 , v is on the branch on v_1 (resp. v_2 , then v_3) containing x_2, x_3 (resp. v_1, v_3 , then v_1, v_2). It follows that v separate v_1, v_2, v_3 in T_2 . The same argument prove the converse property. \square

Proof. [Lemma 4.5] As $T_1 \triangleleft T_2$, the set of vertices of T_1 is included in the one of T_2 . Any internal vertex v_2 of T_2 separates three leaves so, because these leaves are also leaves of T_1 , there is a unique vertex v_1 of T_1 separating them

and it follows that $v_1 = v_2$. Hence T_1 and T_2 have the same set of vertices. From this we deduce that $T_2 \triangleleft T_1$ and it is easy to check that $T_1 = T_2$. \square

4.3.2 Proof of $T_f = T^X$ and $T_f^{-1} \triangleleft T^Y$

Lemma 4.7. *Up to relabeling the internal vertices of T^X , we have $T_f = T^X$.*

Proof. Consider a vertex $v \in \mathcal{T}_f$. We deduce from the definition of X that $v \in T^X$ separates three vertices x_1, x_2 and x_3 which are elements of X . Let us relabel the vertex in T^X separating the same three elements in T^X by v . Then consider a projective chart $M_{t_n} : \mathbb{S} \rightarrow \hat{\mathbb{C}}$ that maps $\Phi_{t_n}(x_1), \Phi_{t_n}(x_2)$ and $\Phi_{t_n}(x_3)$ to $0, 1$ and ∞ .

Take $x \in X$ and suppose that x and x_1 are on the same branch on $v \in \mathcal{T}_f$, then there is a curve $\gamma \subset \partial v$ such that x and x_1 are in the same component of $\mathbb{S} \setminus \gamma$. By construction of \mathcal{T}^X and according to Proposition 4.1, the points $M_{t_n} \circ \Phi_{t_n}(x)$, $M_{t_n} \circ \Phi_{t_n}(x_1)$ and the points $M_{t_n} \circ \Phi_{t_n}(x_2)$, $M_{t_n} \circ \Phi_{t_n}(x_3)$ lies in different components of $\hat{\mathbb{C}}$ minus an annulus whose modulus converges to infinity. Hence

$$M_{t_n} \circ \Phi_{t_n}(x) \rightarrow 0 = M_{t_n} \circ \Phi_{t_n}(x_1),$$

so x and x_1 are in the same branch on $v \in T^X$. As we said in Remark 4.4, the trees T_f and T^X are stable with the same set of leaves, hence Lemma 4.6 and Lemma 4.5 conclude the proof. \square

With the same reasoning we can also show the following:

Lemma 4.8. *After changing the labeling of the internal vertices of T_f^{-1} we have $T_f^{-1} \triangleleft T^Y$.*

We suppose for the rest of this paper that the labeling of the internal vertices of T^X and T_f^{-1} is such that $T_f = T^X$ and $T_f^{-1} \triangleleft T^Y$.

4.3.3 Maps and local degree

Lemma 4.9. *Every edge of T_f^{-1} considered as a subset of T^Y maps onto an edge of T_f considered as a subset of T^Z . All the elements of an edge of T_f^{-1} considered as a subset of T^Y have same degree.*

This lemma is proven in Annexe A.

Let us recall the following result which is a direct consequence of the Argument Principle:

Lemma 4.10. *Let $g : \mathbb{S} \rightarrow \mathbb{S}$ be a branched cover, X be a finite subset of \mathbb{S} containing the critical values of g , Γ be a multicurve on $\mathbb{S} \setminus X$, γ^{-1} be a connected component of $g^{-1}(\Gamma)$ and D be a connected component of $\mathbb{S} \setminus \gamma^{-1}$.*

Then $\gamma := g(\gamma^{-1})$ is the unique curve of Γ such that there exist $x, x' \in X$ in the different connected components of $\mathbb{S} \setminus \gamma$ such that

$$\deg_f(\gamma^{-1}) = |\text{card } D \cap f^{-1}(x) - \text{card } D \cap f^{-1}(x')|,$$

where the cardinals are counted with multiplicity. Moreover this formula works for any couple (x, x') chosen in different connected components of $\mathbb{S} \setminus \gamma$.

In [Arfc](proof of Proposition 3.14), the lemma below is proven by passing to the limit the previous lemma.

Lemma 4.11. *Let $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ be a cover between trees of spheres limit of marked rational maps. Let e be an edge of T^Y adjacent to a vertex v and D denote a connected component of $T^Y \setminus \{e\}$. Then the edge $F(e)$ is the unique edge of T^Z satisfying for a couple $(z, z') \in Z^2$ with z and z' lie in different components of $T^Z \setminus \{F(e)\}$*

$$\deg_{f_v}(i_v(e)) = |\text{card } D \cap F^{-1}(z) - \text{card } D \cap F^{-1}(z')|,$$

where $\deg_{f_v}(i_v(e))$ denotes the local degree of f_v at the attaching point of v and the cardinals are counting the critical points with multiplicity. Moreover this equality holds also for any such couple (z, z') .

Proof. [Theorem 4.3] Lemma 4.7 proved the equality $T_f = T^X$.

First consider a vertex v of \mathcal{T}_f^{-1} . Take any edge e_{-1} between v and $v' \in \mathcal{T}_f^{-1}$. As an edge of \mathcal{T}_f^{-1} , e_{-1} is a closed curve in \mathbb{S} . Denote by D a connected component of $\mathbb{S} \setminus e_{-1}$. According to Lemma 4.10, for any $z, z' \in Z$ lying in the distinct connected components of $\mathbb{S} \setminus f(e_{-1})$, we have $\deg_f(e_{-1}) = |\text{card } D \cap f^{-1}(z) - \text{card } D \cap f^{-1}(z')|$.

According to Lemma 4.8, e_{-1} can be considered as a subset of T^Y and Lemma 4.9 assures that $F(e_{-1})$ is the arc between $F(v)$ and $F(v')$ in T^Z which corresponds to an edge e of T^X . Comparing the formulas given in Lemma 4.11 and Lemma 4.10, as f and F are equal on Y , we deduce that $e = f(e_{-1})$. Hence we proved that

$$F(e_{-1}) = F([v, v']) = [F(v), F(v')] = e = f(e_{-1}).$$

And the assumptions about the local degrees follow the same way. \square

5 Trees and rescaling limits

In this section we prove Theorem 1 and Theorem 2.

Let f be an element of the sequence described in Section 3. According to Remark 3.2, the map τ_f has a periodic vertex v of period $k > 1$, the degree along this cycle is 4 and a non critical branch of \mathcal{T}_f^{-1} maps to a critical one of \mathcal{T}_f by τ_f^k so the corresponding f_v^k is not monomial. Hence Theorem 4.3 proves Theorem 2.

Theorem 1 is a translation of Theorem 2 via the following theorem.

Theorem C ([Arfa]Theorem 2). *Let \mathbf{F} be a portrait, let $(f_n, y_n, z_n)_n \in \text{Rat}_{\mathbf{F}, X}$ and let $(\mathcal{F}, \mathcal{T}^X)$ be a dynamical system of trees of spheres. Suppose that*

$$f_n \xrightarrow[\phi_n^Y, \phi_n^Z]{\triangleleft} \mathcal{F}.$$

If v is a periodic internal vertex in a critical cycle with exact period k , then $f_v^k : \mathbb{S}_v \rightarrow \mathbb{S}_v$ is a rescaling limit corresponding to the rescaling $(\phi_{n,v}^Y)_n$.

In addition, for every v' in the cycle, $(\phi_{n,v'}^Y)_n$ and $(\phi_{n,v}^Y)_n$ are dynamically dependent rescalings.

Appendix A: Compatibility of covers

Here we prove a more general result than Lemma 4.9.

Definition A.1. *We say that a cover between trees of spheres $\mathcal{F}' : \mathcal{T}^{Y'} \rightarrow \mathcal{T}^{Z'}$ is compatible with a cover between trees of spheres $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ of same degree and write $\mathcal{F}' \triangleleft \mathcal{F}$ if*

- $\mathcal{T}^{Y'} \triangleleft \mathcal{T}^Y$,
- $\mathcal{T}^{Z'} \triangleleft \mathcal{T}^Z$, and
- $F' = F|_{Y'}$.

For two vertices v, v' of a combinatorial tree T , we define $[v, v']$ to be the arc between v and v' and $]v, v'[:= [v, v'] \setminus \{v, v'\}$. We define the annulus $]v, v'[[$ to be the connected component of $T \setminus \{v, v'\}$ containing $]v, v'[[$. For a cover between trees of spheres \mathcal{F} we denote by \mathcal{V}_F the set of leaves which are the images of the critical leaves of \mathcal{F} .

Theorem A.2. *Let $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ be a cover between trees of spheres and Z' be a set with at least three elements containing \mathcal{V}_F . There exists a unique cover between trees of spheres $\mathcal{F}' : \mathcal{T}^{F^{-1}(Z')} \rightarrow \mathcal{T}^{Z'}$ such that $\mathcal{F}' \triangleleft \mathcal{F}$.*

Proof. We define $\mathcal{T}^{Z'}$ to be the tree of sphere whose set of leaves is Z' , whose internal vertices are the vertices of T^Z separating at least three elements of Z' and whose edges and their attaching points are such that $\mathcal{T}^{Z'} \triangleleft \mathcal{T}^Z$. We define $Y' := F^{-1}(Z')$ and $\mathcal{T}^{Y'}$ whose vertices are the preimage by F of the one of $T^{Z'}$. Clearly the internal vertices of $T^{Y'}$ are the vertices of T^Y who separate at least three elements of Y' . We define again the edges and their attaching points of $\mathcal{T}^{Y'}$ to be such that $\mathcal{T}^{Y'} \triangleleft \mathcal{T}^Y$. We just have to prove that $F' := F|_{\mathcal{T}^{Y'}}$ is a combinatorial tree map and that the corresponding $\mathcal{F}' := \mathcal{F}|_{\mathcal{T}^{Y'}}$ is a cover between trees of spheres.

Consider an edge of $T^{Y'}$ between two vertices v and v' . As $\mathcal{V}_F \subset Z'$, the set Y' contains all of the critical leaves of F and the annulus $]v, v'[[$ does not contain any critical leaf. Then the result follows from Lemma A.3 below. \square

Lemma A.3. *If Suppose that $[v, v']$ is an arc and that $]v, v'[[$ does not contain any critical leaf, then all of the elements of $[v, v']$ have same degree and $F([v, v']) = [F(v), F(v')]$.*

Proof. The critical internal vertices form paths between critical leaves (see [Arfb]). Hence, as $]v, v'[[$ does not contain critical leaves, its only critical vertices can be on $]v, v'[[$. It follows that for any $v'' \in]v, v'[[$, the map $f_{v''}$ has exactly two critical points, so these ones have same degree and all the $f_{v''}$ have same degree.

The image of $[v, v']$ is a connected graph that contains $F(v)$ and $F(v')$ so $[F(v), F(v')] \subset F([v, v'])$. Suppose that $F([v, v'])$ has an end $F(w)$ different than $F(v)$ and $F(v')$. Then $F(w)$ has an unique adjacent edge in $F([v, v'])$ whose attaching point is thus the image of the two only critical points of $f_{v''}$ which is no possible, so we have a contradiction. \square

For every cover \mathcal{F} , let us denote by $\Pi_{\mathbf{F}, \mathbf{F}'}(\mathcal{F})$ the cover \mathcal{F}' associated to \mathcal{F} by the previous theorem. In fact we have proven the following result:

Theorem A.4. *The application $\Pi_{\mathbf{F}, \mathbf{F}'} : \overline{\mathbf{rev}}_{\mathbf{F}} \rightarrow \overline{\mathbf{rev}}_{\mathbf{F}'}$ is continuous.*

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